

Fidelity optimization for holonomic quantum gates in dissipative environments

Daniele Parodi,^{1,2} Maura Sassetti,^{1,3} Paolo Solinas,^{1,2} Paolo Zanardi⁴, Nino Zanghì^{1,2}

¹ Dipartimento di Fisica, Università di Genova, Via Dodecaneso 33, 16146 Genova, Italy

² Istituto Nazionale di Fisica Nucleare (Sezione di Genova), Via Dodecaneso 33, 16146 Genova, Italy

³ INFN-CNR Lamia, Via Dodecaneso 33, 16146 Genova, Italy

⁴ Institute for Scientific Interchange, Viale Settimio Severo 65, 10133 Torino, Italy

(Dated: February 1, 2008)

We analyze the performance of holonomic quantum gates in semiconductor quantum dots, driven by ultrafast lasers, under the effect of a dissipative environment. The environment is modeled as a thermal bath of oscillators linearly coupled with the electron states of the quantum dot. Standard techniques make the problem amenable to a numerical treatment and allow one to determine the *fidelity* as a function of all the relevant physical parameters. As a consequence of our analysis, we show that the disturbance of the environment can be (approximately) suppressed and the performance of the gate optimized—provided that the thermal bath is purely superohmic. We conclude by showing that such an optimization is impossible for ohmic environments.

PACS numbers: 03.67.Lx

I. INTRODUCTION

In the last years holonomic quantum computation (HQC) has proved to be a viable and promising approach to quantum information processing and quantum computation [1]. According to this approach, quantum information is encoded in an n -fold degenerate eigenspace of a family of quantum Hamiltonians depending on dynamically controllable parameters. Recently, concrete proposals for quantum computation have been put forward, for both Abelian [2] and non-Abelian holonomies [3, 4]. The so-constructed gates depends only on global geometrical feature, e.g., the solid angle spanned in the parameter space, and because of this, it is believed that they are robust against errors affecting the physical parameters controlling the gates themselves (e.g., laser pulses). This expectation has been confirmed by recent investigations [5].

An important open problem is whether holonomic quantum gates are stable under the effect of the environment [6]. In this paper we argue that holonomic gates have indeed a good performance when the effect of the environment is taken into account. We show this on the basis of a simple and idealized model which covers situations describing electron states in quantum dots ranging from excitons [4] to optically active spin-degenerate ones [7] (for a related investigation about environmental effects on semiconductor based quantum gates see Ref. [8]). For such gates, the main source of dissipation is due to phonons with *superohmic* spectral density [9]. It turns out that these gates manifest a rich structure when the “control parameters” are changed. By varying in a suitable way the adiabatic time, the *superohmic* effect can be minimized and suppressed. For completeness, we have extended our analysis to an ohmic environment.

In Sec. II, we review the HQC model, introduce a model for the dissipative environment and write the master equation to solve. In Sec. III, we describe the computer simulations and specify the kind of environment

we consider. From the numerical results we deduce non-trivial behavior of the *fidelity* (the gate performance estimator) which allows us to minimize the decoherence effects of a *superohmic* environment. We show that analogous minimization cannot be done with other kinds of environment (*ohmic*).

II. HQC IN A DISSIPATIVE ENVIRONMENT

The physical system we consider is constituted by three degenerate (or quasi-degenerate) states ($|+\rangle$, $|-\rangle$, and $|0\rangle$) optically connected to another state $|G\rangle$. Every degenerate state is separately addressed by polarization or frequency selection with a laser. This model describes various quantum systems interacting with a laser radiation field; here, we deal with quantum states in semiconductor quantum dots such as excitons [4] and spin-degenerate electron states [7].

The (approximate) Hamiltonian modeling the effect of the laser on the system is (for simplicity, $\hbar = 1$) [4]

$$H_0(t) = \sum_{j=+,-,0} (\epsilon|j\rangle\langle j| + e^{-i\epsilon t}\Omega_j(t)|j\rangle\langle G|) + h.c. , \quad (1)$$

where $\Omega_j(t)$ are the time dependent Rabi frequencies depending on controllable parameters, such as the phase and intensity of the lasers, and ϵ is the energy of the degenerate electron states. The Rabi frequencies can be modulated within the adiabatic time t_{ad} (which coincides with the gating time) to produce a loop in the parameter space and thereby realize the periodic condition $H_0(t_{ad}) = H_0(0)$.

It should be observed that different loops in the parameter space produce different holonomic operators. Here we consider two different sets of Rabi frequencies, that is, two different choices of the time dependent functions $\Omega_j(t)$ in Eq. (1). According to the holonomic approach, the corresponding unitary evolutions occurring in the adiabatic time define the unitary

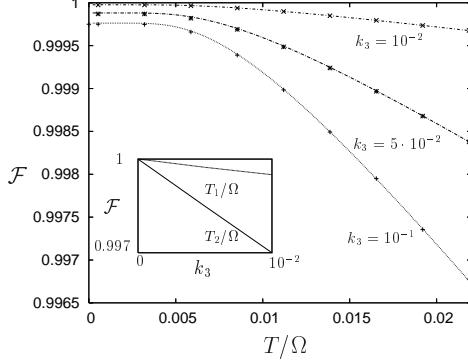


FIG. 1: Fidelity \mathcal{F} for gate 1 as a function of temperature for different k_3 (expressed in $(\text{meV})^{-2}$) in the presence of a *superohmic* environment. The points are the computer simulation results and the curves have the form $\mathcal{F} = 1 - \sum_{j=\pm} \eta_j \Gamma^j t_{ad}$ ($\eta_- = 3 \cdot 10^{-2}$ and $\eta_+ = 0.7$). Inset: Fidelity \mathcal{F} as a function of k_3 for two different temperatures ($T_1/\Omega = 1.6 \cdot 10^{-2}$ and $T_2/\Omega = 5 \cdot 10^{-3}$). Parameters: $\epsilon = 1 \text{ eV}$, $\Omega = 25 \text{ meV}$, $t_{ad} = 7.5 \text{ ps}$ and $\omega_c = 0.5 \text{ meV}$.

transformations associated with the holonomic quantum gates [4]. We shall hereafter refer to the gates defined by these two different sets of Rabi frequencies (which form a complete set of single qubit gates) as ‘‘gate 1’’ and ‘‘gate 2’’; they correspond, respectively, to the unitary operators $U_1 = e^{i\frac{\pi}{4}|+\rangle\langle+|}$ and $U_2 = e^{i\frac{\pi}{2}\sigma_y}$ (where $\sigma_y = i(|+\rangle\langle-| - |-\rangle\langle+|)$).

The Hamiltonian (1) has four eigenstates: two eigenstates $|B_{\pm}\rangle$ with time-dependent eigenvalues ϵ_{\pm} (called *bright states*) and two eigenstates $|D_{1,2}\rangle$ with constant and degenerate eigenvalue ϵ (called *dark states*). To construct a complete set of holonomic quantum gates it is sufficient to restrict Rabi frequencies $\Omega_j(t)$ such that the norm Ω of the vectors $\{\Omega_j(t)\}$ is time independent. Under this condition, it can be easily shown that the two dark states have energy ϵ and the two bright states have time independent energies

$$\epsilon_{\pm} = (\epsilon \pm \sqrt{\epsilon^2 + 4\Omega^2})/2. \quad (2)$$

The adiabatic condition is simply $\Omega_{ad} \gg 1$.

The environment is described as an ensemble of harmonic oscillators linearly coupled to the system [10], with total Hamiltonian

$$H = H_0(t) + \sum_{\alpha=1}^N \left(\frac{p_{\alpha}^2}{2m_{\alpha}} + \frac{1}{2}m_{\alpha}\omega_{\alpha}^2x_{\alpha}^2 + c_{\alpha}x_{\alpha}A \right) \quad (3)$$

The interaction should break the degeneracy of the degenerate states, a condition that is easily fulfilled by assuming the operator A to be of the form $A = \text{diag}(0, 1, 0, -1)$ in the basis $|G\rangle$, $|+\rangle$, $|0\rangle$, and $|-\rangle$. (Eq. (3) includes implicitly the standard renormalization term [9].)

We now consider the time evolution of the reduced density matrix of the system, determined by the Hamiltonian

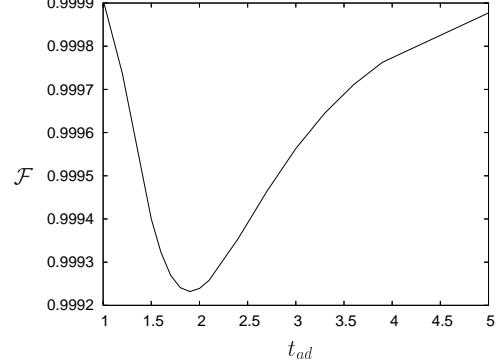


FIG. 2: Fidelity \mathcal{F} for gate 1 subject to *superohmic* spectral density as a function of t_{ad} ($T/\Omega = 8.5 \cdot 10^{-3}$, $k_3 = 10^{-1}$ ($\text{meV})^{-2}$ and $\Omega t_{ad} = 280$). The adiabatic time is normalized to 5 ps. Parameters as in Figure 1 except t_{ad} .

(3). We rely on the standard methods of the ‘‘master equation approach,’’ according to which the effect of the environment is considered in the Born approximation, and the environment is assumed to be at each time in its own thermal equilibrium state at temperature T . One has [9]

$$\dot{\tilde{\rho}}(t) = - \int_0^t d\tau \{ g(\tau) [\tilde{A}\tilde{A}'\tilde{\rho}(t-\tau) - \tilde{A}'\tilde{\rho}(t-\tau)\tilde{A}] + g(-\tau) [\tilde{\rho}(t-\tau)\tilde{A}'\tilde{A} - \tilde{A}\tilde{\rho}(t-\tau)\tilde{A}'] \} \quad (4)$$

where, $\tilde{\rho}(t)$ denotes the time evolution of the reduced density matrix of the system in the interaction picture, e.g., $\tilde{\rho}(t) = U^{\dagger}(t, 0)\rho U(t, 0)$, where $U(t, 0) = T(\exp -i \int_0^t dt' H_0(t'))$ and T is the time ordered product. In the above equation \tilde{A} and \tilde{A}' stand for $\tilde{A}(t)$ and $\tilde{A}(t-\tau)$ (again, the superscript tilde means time evolution in the interaction picture). In Eq. (4) the effect of the environment is included in the function (for simplicity, $k_B = 1$)

$$g(\tau) = \int_0^{\infty} J(\omega) [\coth(\frac{\omega}{2T}) \cos(\omega\tau) - i \sin(\omega\tau)] d\omega \quad (5)$$

where the spectral density $J(\omega)$ is defined in a standard way in terms of oscillator parameters in Eq. (3) and its typical behavior, for physical environments in the low frequency regimes, is proportional to ω^s , with $s \geq 1$ [9]; the asymptotic decay of the real part of $g(\tau)$ defines the characteristic memory time τ_E of the environment.

We solve Eq. (4) for an environment with memory time τ_E faster than the time scale of the variation of the density matrix τ_D (estimated self-consistently), so that $\tilde{\rho}(t-\tau) \approx \tilde{\rho}(t)$ (Markov approximation). In this approximation, Eq. (4) simplifies and assumes a convenient form—for the numerical analysis we wish to perform—in the Schrödinger representation:

$$\dot{\rho}(t) = -i[H_0(t), \rho(t)] - \mathcal{L}(\rho) \quad (6)$$

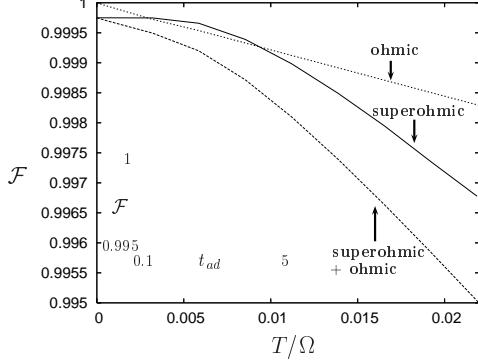


FIG. 3: Fidelity \mathcal{F} for gate 1 subject to *superohmic* (dashed line), *ohmic* (dot line) and both (solid line) spectral densities as a function of T/Ω . Inset: Fidelity \mathcal{F} for *superohmic* plus *ohmic* environment as a function of t_{ad} with $\Omega t_{ad} = 280$ ($k_1 = 4 \cdot 10^{-4}$ and $k_3 = 10^{-1}(\text{meV})^{-2}$). Parameters as in Figure 1.

with

$$\mathcal{L}(\rho) = \int_0^\infty d\tau \{ g(\tau) [AA''\rho(t) - A''\rho(t)A] + g(-\tau) [\rho(t)A''A - A\rho(t)A''] \}, \quad (7)$$

where $A'' = U^\dagger(t - \tau, t)AU(t - \tau, t)$, and, in the adiabatic approximation, $U(t - \tau, t) \approx \exp(i\tau H_0(t))$. As usual, the upper extreme of integration is extended to infinity because the evolution time is much longer than τ_E .

III. COMPUTER SIMULATIONS AND RESULTS

In order to estimate how the environment affects the performance of the *ideal* holonomic gates determined by the “dissipation-free” Hamiltonian $H_0(t)$, we use the standard notion of *fidelity*

$$\mathcal{F} = \sqrt{\langle \psi_{id}(t_{ad}) | \rho(t_{ad}) | \psi_{id}(t_{ad}) \rangle} \quad (8)$$

where $|\psi_{id}(t_{ad})\rangle$ is the state in which the initial (pure) state evolves, in the adiabatic time t_{ad} , under the action only of $H_0(t)$, while $\rho(t_{ad})$ is the solution of Eq. (6), computed at the same time, and for the *same* initial (pure) state. In order to avoid dependence on the latter we have taken a suitable average on the initial states. We make a sampling of the initial logical states (combination of the logical states $|+\rangle$ and $|-\rangle$) on the Bloch sphere. We add the possibility of an error in the preparation of the initial state with the population of the *non-logical* state $|0\rangle$. This can be due to the imprecise control of the lasers. The initial state has the form $\alpha|+\rangle + \beta|-\rangle + \eta|0\rangle$ with $|\alpha|^2 + |\beta|^2 + |\eta|^2 = 1$ and $|\eta|^2 = 0.1$. In the following, with a slight abuse of notation, we shall denote by the same symbol \mathcal{F} this averaged fidelity.

An essential ingredient of our analysis is the spectral density entering in Eq. (5). For the electronic states in

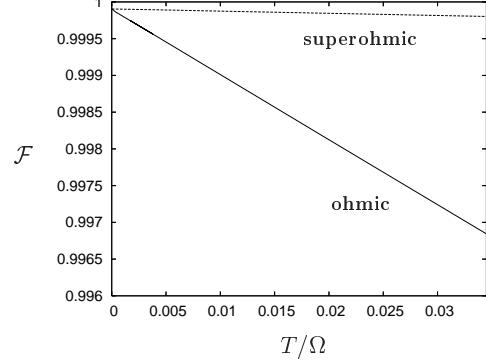


FIG. 4: Fidelity for two qubit gate subject to *superohmic* and *ohmic* spectral density as a function of T/Ω for $k_1 = 10^{-4}$ and $k_3 = 10^{-2}(\text{meV})^{-2}$. Parameters as in Figure 1 except $t_{ad} = 0.8$ ns and $\Omega = 0.2$ meV.

quantum dots the decoherence effects are principally due to phonons. Single phonon processes are described by *superohmic* spectral densities with $J(\omega) = k_3\omega^3 e^{-(\omega/\omega_c)^2}$ [11]. The high frequency cut-off ω_c is due to the planar confinement in the quantum dot. The adimensional coupling constant k_3 allows the description of different kinds phonon-carrier interactions in semiconductor materials including deformation potential, piezoelectric and spin-orbit [12].

The results of numerical simulations for the fidelity \mathcal{F} of gate 1, as a functions of the temperature are shown in Fig. 1: at low temperature the fidelity is constant and decreases linearly as the temperature increases. To understand how this behavior comes about it is convenient to express the dissipative part of $\mathcal{L}(\rho)$ in Eq. (6) (in the dark-bright state basis) as [9]:

$$\sum_{kl} (\delta_{lm} \sum_r \Gamma_{nrrk}^+ + \delta_{nk} \sum_r \Gamma_{lrrk}^- - \Gamma_{lmnk}^- - \Gamma_{lmnk}^+) \rho_{kl}.$$

Here, $\Gamma_{lmnk}^\pm = \Gamma_{nk}^\pm K_{lmnk}$, with K_{lmnk} depending only on laser parameters and

$$\Gamma_{nk}^\pm(\omega_{nk}) = J(\omega_{nk}) \left(\coth\left(\frac{\omega_{nk}}{2T}\right) \mp 1 \right) \quad (9)$$

are the transition rates between n and k states due to phonons. In passing, we note that these are indeed the rates that could be guessed by a straightforward application of Fermi’s golden rule to the interaction terms of Eq. (3). Finally, observe that the frequencies $\omega_{nk} = \epsilon_n - \epsilon_k$ represent the energy differences in the dark-bright space, i.e., $\epsilon_n = \epsilon, \epsilon_\pm$, where ϵ_\pm are given by Eq. (2); for $\epsilon \gg \Omega$, $\omega_{nk} = 0, \epsilon, \Omega^2/\epsilon$.

Equation (9) shows that, with *superohmic* spectral density, the only relevant transition is the one with $\omega_{nk} = \Omega^2/\epsilon$ giving the transition rates $\Gamma^\pm(\Omega^2/\epsilon)$. In fact, for $\omega_{nk} = \epsilon$ the gaussian cut-off with $\omega_c \ll \epsilon$ produces negligible rates, and for degenerate dark state ($\omega_{nk} = 0$) the rates vanish. These considerations, together with

the explicit form of the rates given by Eq. (9) provide a compelling explanation of the temperature behavior of the fidelity in Fig.1. Moreover, we have found that the numerical results are fitted by means of the function

$$\mathcal{F} = 1 - t_{ad} \sum_{j=\pm} \eta_j \Gamma^j \quad (10)$$

(where η_j are two real parameters). This behavior is also manifest by considering the fidelity as a function of the coupling parameter k_3 (see inset in Fig. 1). A similar dependence of the fidelity on the transition rate has been found in Ref. [8]. Note that in that case the authors have only a single transition process (absorption to higher states or emission to lower states) while we need to take into account both absorption and emission processes for the transition to the higher state. This is due to the different master equation solved. In fact, they solve a strictly second order master equation, while our eq. (6) is “self-consistent”. The similarity in the results is derived from the small value of the coupling constant k_3 but for higher values the two approximations diverge from each other and the numerical results cannot be fitted by such an elementary function.

Eq. (10) is particularly important in that it allows one to predict how the fidelity behaves when the parameters of the system are modified. To this end, first of all note that, by keeping the adiabatic parameter constant ($\Omega t_{ad} = \alpha = \text{const}$), the rates Γ^\pm become a non-trivial function of the adiabatic time. Then, by writing the rates explicitly, one obtains $\Gamma^\pm \propto 1/t_{ad}^6 \coth(\alpha^2/(\epsilon t_{ad}^2 T)) \exp(-(\alpha^2/(t_{ad}^2 \epsilon \omega_c))^2)$. Thus, it follows from Eq. (8) that the fidelity should have a pronounced minimum as a function of t_{ad} . This behavior is confirmed by the computer simulations presented in Fig. 2. By varying Ω and t_{ad} (e.g., by acting on the lasers) the position of the fidelity minimum in Fig. 2 can be shifted and the effect of *superohmic* environment can be suppressed. It seems to us that this is an interesting result.

Before discussing the limitations of this result, we would like to comment on the approximations upon which our analysis relies. As we have already anticipated, in Eq. (6) the Markov approximation is appropriate when the memory time τ_E is small with respect to the time scale of variation of the density matrix τ_D . Eq. (5) leads to the estimate $\tau_E \approx 1/(2\pi T)$; while Eq. (9), for a *superohmic* spectral density and $T \ll \Omega^2/\epsilon$, leads to $\tau_D = (\epsilon/\Omega^2)^3/k_3$. Note that the conditions of validity of the Markov approximation, $\tau_E < \tau_D$, readily translate into a temperature regime, namely $T > T_M = k_3(\Omega^2/\epsilon)^3$. With our choice of parameters, we have $1.2 \cdot 10^{-4} \leq T_M/\Omega \leq 1.2 \cdot 10^{-3}$ (depending on k_3 value), which, in our simulations, is a very low temperature.

Since the possibility of suppressing the *superohmic* effects is indeed surprising, one may wonder whether a similar possibility arises for more general environments, e.g. for ohmic environments. Though ohmic environments are typical of baths of conduction electrons [9], even for

phonon baths, which are typically *superohmic*, it seems possible that the spectral density contains an ohmic part; this is presumably due to higher order contributions such as two phonon processes [13].

Be that as it may, we found it interesting to extend our analysis to an environment with the spectral density $J(\omega) = k_1 \omega e^{-(\omega/\omega_c)^2}$ with $k_1 \ll 1$. As is easily seen, Eq. (9) for ohmic rates leads to completely different results. This is due to the presence, in the ohmic case, of transitions between degenerate states which are absent in the case of a superohmic environment. This difference has striking consequences: the transitions between degenerate states give contribution to the rates (9) which are linear in T (while the transitions between non-degenerate states have the same temperature behavior as the *superohmic* case). This difference is confirmed by the computer simulations in Fig. 3, which shows the fidelity for the superohmic and ohmic environments, and the sum of the two contributions as a function of T . This curve clearly shows that the presence of an ohmic environment changes dramatically the fidelity behavior, whence it follows the impossibility of extending to the ohmic case the results previously obtained changing the adiabatic time. This conclusion becomes very clear if one compares Fig. 2 with the inset in Fig. 3. It is not possible anymore to optimize the fidelity by changing the parameters.

Before concluding, we briefly mention three points (see [14] for a thorough discussion). First, the computer simulation for gate 2 confirms the results found for gate 1. Second, our analysis extends (almost straightforwardly) to the two-qubit gate proposed in Ref. [4]: Fig. 4 shows the preliminary results for the two-qubit gate (for a non random initial state) as a function of temperature and coupling constant for *superohmic* and *ohmic* environments; the behavior of the fidelity is analogous to that of the one qubit case (Fig. 3). Third, for the ohmic case, a careful study of the (relatively) high temperature behavior shows the limitations of the Lindblad approximation for the reduced dynamics.

IV. CONCLUSIONS

To sum up, the upshot of our analysis is twofold. The good news is that it is possible to optimize the fidelity for the kind of environment which is usually considered for electron states in quantum dots—a superohmic environment caused by electron-phonon interactions. The bad news is that such optimization does not go through an ohmic environment, e.g., produced by the *same* superohmic phonon bath through two phonon processes. Thus, particular attention should be paid in modeling the environment, since the presence of a weak ohmic environment dramatically changes the holonomic gate performance. For these reasons it is crucial to dispose of experimental investigations on the nature of the environmental spectral densities in semiconductor quantum dots.

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